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Existence results for a variable exponent elliptic problem *via* topological method

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Abstract

In this paper, existence, localization and uniqueness results of solutions to elliptic Dirichlet boundary value problems are established. The approach is based on the nonlinear alternative of Leray-Schauder, the Brouwer fixed point theorem and the Galerkin method.

MSC: 35J60; 47H10

Keywords: $p(x)$ -Laplacian; Leray-Schauder principle; fixed point; Galerkin method

1 Introduction

In this paper, we consider the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a nonempty bounded open set with smooth boundary $\partial\Omega$, $p = p(x) \in C_+(\overline{\Omega})$ with $1 < p^- := \min_{\overline{\Omega}} p(x) \leq p^+ := \max_{\overline{\Omega}} p(x) < +\infty$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$ -Laplacian and becomes p -Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$ -Laplacian possesses more complicated nonlinearities than the p -Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with a variable exponent growth condition has received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to a problem of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electro-magnetic field [1, 2]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal baro-tropic gas through a porous medium [3, 4]. Another field of application of equations with variable exponent growth conditions is image processing [5]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [6–11] for an overview of and references on this subject, and to [12–21] for the study of the $p(x)$ -Laplacian equations and the corresponding variational problems.

In recent years, many authors have studied the existence of solutions for problem (1.1) from several points of view and with different approaches (see, for example, [18–20]). A useful method for the investigation of solutions to semilinear problems is based on the Leray-Schauder continuation principle, or equivalently, on Schaefer's fixed point theorem. For example, in [22] this method was used for solutions in Hölder spaces, while in [23], solutions were found in Sobolev spaces.

The aim of this paper is to present new existence, localization and uniqueness results for solutions to problem (1.1) under suitable conditions on the nonlinearity f . Our approach is based on regularity results for the solutions of linear Dirichlet problems, the nonlinear alternative of Leray-Schauder (see [24]), the Brouwer fixed point theorem (see [25]) and the Galerkin method. We notice that our partial results of the present paper are motivated by the papers [26] and [27] where the authors have obtained some results for semilinear and quasilinear elliptic boundary value problems, respectively. By the Galerkin method, we also establish the results of existence and uniqueness of a solution for problem (1.1). We also would like to point out that the proof of Theorem 3 of [27] is wrong since $(-\Delta_p)^{-1}$ is not a linear operator. In this paper, we give a key lemma that can be used to overcome this difficulty.

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we shall use a nonlinear alternative of Leray-Schauder to prove the existence of solutions for problem (1.1). In Section 4, by the Galerkin method, we shall establish the results of existence and uniqueness of a solution for problem (1.1).

2 Preliminaries

In order to discuss problem (1.1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call a variable exponent Sobolev space. Firstly, we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [7]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω .

Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$$

and

$$L^{p(x)}(\Omega) = \left\{ u \in \mathbf{S}(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.1 (See [7]) *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$. For any $u \in L^{p(x)}(\Omega)$, then*

- (1) for $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;
- (2) $|u|_{p(x)} < 1$ ($= 1; > 1$) $\Leftrightarrow \rho(u) < 1$ ($= 1; > 1$);
- (3) if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
- (4) if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
- (5) $\lim_{k \rightarrow +\infty} |u_k|_{p(x)} = 0 \iff \lim_{k \rightarrow +\infty} \rho(u_k) = 0$;
- (6) $\lim_{k \rightarrow +\infty} |u_k|_{p(x)} = +\infty \iff \lim_{k \rightarrow +\infty} \rho(u_k) = +\infty$.

In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant C_0 such that

$$|u|_{L^{p(x)}(\Omega)} \leq C_0 |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ in the space $W_0^{1,p(x)}(\Omega)$. We shall use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.2 (1) [17, Theorem 4.1] *If f satisfies the sub-critical growth condition*

$$|f(x, t)| \leq c(1 + |t|^{q(x)-1}), \quad \forall x \in \Omega, \forall t \in \mathbb{R}, \tag{2.1}$$

where $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$, $\forall x \in \overline{\Omega}$, where $p^*(x)$ is the Sobolev critical exponent ($p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ and $p^* = +\infty$ if $p(x) \geq N$), then $u \in L^\infty(\Omega)$ for every weak solution u of (1.1).

(2) [17, Theorem 4.4] *Let $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ be a solution of (1.1). If the function p is log-Hölder continuous on $\overline{\Omega}$, i.e., there is a positive constant H such that*

$$|p(x) - p(y)| \leq \frac{H}{-\log|x - y|} \quad \text{for } x, y \in \overline{\Omega} \text{ with } |x - y| \leq \frac{1}{2}, \tag{2.2}$$

then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

(3) [21, Theorem 1.2] *If in (2), the condition (2.2) is replaced by that p is Hölder continuous on $\overline{\Omega}$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

From [20], we know that, for a given $h(x) \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, where $q(x)$ satisfies (2.1), the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique solution $u \in W_0^{1,p(x)}(\Omega)$. We denote by $K(h) := u$ the unique solution. K is called the solution operator for problem (2.3). It is well known that the solution operator K is increasing (see Remark 2.1 of [18]). From the Proposition 2.2 and the embedding theorems, we can obtain the properties of the solution operator K as follows.

Proposition 2.3 (1) (See [20]) The mapping $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \rightarrow W_0^{1,p(x)}(\Omega)$ is continuous. Moreover, the mapping $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \rightarrow L^{q(x)}(\Omega)$ is completely continuous since the embedding $W_0^{1,p(x)}(\Omega) \rightarrow L^{q(x)}(\Omega)$ is compact.

(2) (See [17]) If p is log-Hölder continuous on $\bar{\Omega}$, then the mapping $K : L^\infty(\Omega) \rightarrow C^{0,\alpha}(\bar{\Omega})$ is bounded, and hence the mapping $K : L^\infty(\Omega) \rightarrow C(\bar{\Omega})$ is completely continuous.

(3) (See [21]) If p is Hölder continuous on $\bar{\Omega}$, then the mapping $K : L^\infty(\Omega) \rightarrow C^{1,\alpha}(\bar{\Omega})$ is bounded, and hence the mapping $K : L^\infty(\Omega) \rightarrow C^1(\bar{\Omega})$ is completely continuous.

We note that the method of [26] cannot be directly used in this paper since K is not a linear operator. So, we give a key lemma that will be used in Section 3 to overcome this difficulty.

Lemma 2.1 Let M be a positive constant and $h(x) \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, then there exists one point $\xi \in \Omega$ such that

$$K(Mh) = M^{\frac{1}{p(\xi)-1}} K(h),$$

where K is the solution operator for problem (2.3).

Proof We assume that u is a solution of problem (2.3), then we have $u = K(h)$. From (2.3), we can also show that

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx = \int_{\Omega} h(x)u(x) dx. \tag{2.4}$$

By (2.4) and mean value theorem, for any positive constant C , we can show that there exists one point $\xi \in \Omega$ such that

$$\begin{aligned} \int_{\Omega} \frac{1}{p(x)} |\nabla(Cu)|^{p(x)} dx &= C^{p(\xi)} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &= C^{p(\xi)} \int_{\Omega} h(x)u(x) dx \\ &= \int_{\Omega} C^{p(\xi)-1} h(x)Cu(x) dx, \end{aligned}$$

that is to say,

$$K(C^{p(\xi)-1}h) = CK(h).$$

Let $M = C^{p(\xi)-1}$, then $C = M^{\frac{1}{p(\xi)-1}}$. □

3 Existence of a solution via the alternative of Leray-Schauder

Here and in the sequel, E will denote the space

$$C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$$

endowed with the sup-norm

$$\|u\|_0 = \sup_{x \in \bar{\Omega}} |u(x)|.$$

Now, we state an existence and localization principle for problem (1.1).

Theorem 3.1 (1) Assume that f satisfies (2.1), $p(x) \in C_+(\overline{\Omega})$ and there is a constant $r > 0$, independent of $\lambda > 0$, with

$$|u|_{L^{q(x)}(\Omega)} \neq r, \tag{3.1}$$

for any solution $u \in W_0^{1,p(x)}(\Omega)$ to

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

and for each $\lambda \in (0, 1)$. Then the boundary value problem (1.1) has at least one solution $u \in W_0^{1,p(x)}(\Omega)$ with $|u|_{L^{q(x)}(\Omega)} \leq r$.

(2) Assume that $p(x)$ is log-Hölder continuous and there is a constant $r > 0$, independent of $\lambda > 0$, with

$$\|u\|_0 \neq r, \tag{3.3}$$

for any solution $u \in C^{0,\alpha}(\overline{\Omega})$ to (3.2) and for each $\lambda \in (0, 1)$. Then the boundary value problem (1.1) has at least one solution $u \in C^{0,\alpha}(\overline{\Omega})$ with $\|u\|_0 \leq r$.

(3) Assume that $p(x)$ is Hölder continuous and there is a constant $r > 0$, independent of $\lambda > 0$, with (3.3) for any solution $u \in C^{1,\alpha}(\overline{\Omega})$ to (3.2) and for each $\lambda \in (0, 1)$. Then the boundary value problem (1.1) has at least one solution $u \in C^{1,\alpha}(\overline{\Omega})$ with $\|u\|_0 \leq r$.

We note that Theorem 3.1 not only guarantees the existence of a solution, but also gives information about its localization. Since the proofs of Theorem 3.1(1)-(3) are identical, we shall just prove Theorem 3.1(3). Firstly, we recall the following well-known results:

Lemma 3.1 (Nonlinear alternative of Leray-Schauder, [24]) Let $B[0, r]$ denote the closed ball in a Banach space E with radius r , and let $T : B[0, r] \rightarrow E$ be a compact operator. Then either

- (i) the equation $\lambda Tu = u$ has a solution in $B[0, r]$ for $\lambda = 1$, or
- (ii) there exists an element $u \in E$ with $\|u\|_E = r$ satisfying $\lambda Tu = u$ for some $0 < \lambda < 1$.

Proof of Theorem 3.1(3) According to Proposition 2.3, the operator K from $L^\infty(\Omega)$ to $C^1(\overline{\Omega})$ is well defined and compact. We shall apply the nonlinear alternative of Leray-Schauder to E and to the operator $T : E \rightarrow E$, with $Tu = KFu$, where $F : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ is given by $(Fu)(x) = f(x, u(x))$. On the other hand, it is clear that the fixed points of T are the solutions of problem (1.1). Now the conclusion follows from Lemma 3.1 since condition (ii) is excluded by hypothesis. \square

Theorem 3.1 immediately yields the following existence and localization results.

Corollary 3.1 (1) Assume that f satisfies (2.1), $p(x) \in C_+(\overline{\Omega})$ and there exist nonnegative continuous functions $a(x)$, $b(x)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such

that

$$|f(x, u)| \leq a(x)\psi(|u|) + b(x), \quad \forall (x, u) \in \Omega \times \mathbb{R}. \tag{3.4}$$

Suppose, in addition, that there exists a real number $r > 0$ such that

$$r \geq (\psi(r) + 1)^{\frac{1}{p-1}} |K(c(x))|_{L^{q(x)}(\Omega)}, \tag{3.5}$$

where $c(x) = \max\{a(x), b(x)\} = \frac{1}{2}(a(x) + b(x) + |a(x) - b(x)|)$. Then the boundary value problem (1.1) has at least one solution in $W_0^{1,p(x)}(\Omega)$ with $|u|_{L^{q(x)}(\Omega)} \leq r$.

(2) Assume that $p(x)$ is log-Hölder continuous and (3.4) holds. Suppose, in addition, that there exists a real number $r > 0$ such that

$$r \geq (\psi(r) + 1)^{\frac{1}{p-1}} \|K(c(x))\|_0. \tag{3.6}$$

Then the boundary value problem (1.1) has at least one solution in $C^{0,\alpha}(\overline{\Omega})$ with $\|u\|_0 \leq r$.

(3) Assume that $p(x)$ is Hölder continuous and (3.4), (3.6) hold. Then the boundary value problem (1.1) has at least one solution in $C^{1,\alpha}(\overline{\Omega})$ with $\|u\|_0 \leq r$.

Since the proofs of Corollary 3.1(1)-(3) are identical, we shall just prove Corollary 3.1(3).

Proof of Corollary 3.1(3) In order to apply Theorem 3.1(3), we have to show that condition (3.3) holds true for any solution $u \in C^{1,\alpha}(\overline{\Omega})$ to (3.2). Assume $u \in C^{1,\alpha}(\overline{\Omega})$ is any solution to (3.2) for some $\lambda \in (0, 1)$ with $\|u\|_0 = r$. Then

$$u = \lambda Tu = \lambda KFu.$$

Furthermore, for all $x \in \overline{\Omega}$, by the monotonicity of K and Lemma 2.1, we have

$$\begin{aligned} |u| &= \lambda |KFu(x)| \\ &\leq \lambda |K(a(x)\psi(|u|) + b(x))| \\ &\leq \lambda |K(a(x)\psi(r) + b(x))| \\ &\leq \lambda |K(c(x)(\psi(r) + 1))| \\ &\leq \lambda (\psi(r) + 1)^{\frac{1}{p(x)-1}} |K(c(x))| \\ &\leq \lambda (\psi(r) + 1)^{\frac{1}{p(x)-1}} \|K(c(x))\|_0 \\ &\leq \lambda (\psi(r) + 1)^{\frac{1}{p-1}} \|K(c(x))\|_0. \end{aligned}$$

Taking the supremum in the above inequality, we obtain

$$\|u\|_0 \leq \lambda (\psi(r) + 1)^{\frac{1}{p-1}} \|K(c(x))\|_0.$$

Therefore, $r \leq \lambda r < r$ since $\lambda \in (0, 1)$ and $\|u\|_0 = r$. This is a contradiction. □

We note that condition (3.1) can be satisfied under some suitable conditions.

Theorem 3.2 *Suppose that $p(x) \in C_+(\overline{\Omega})$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and satisfies*

$$\lim_{|s| \rightarrow +\infty} \frac{f(x, s)}{s^{p(x)-1}} = 0, \tag{3.7}$$

uniformly with respect to $x \in \Omega$. Then the boundary value problem (1.1) has a solution $u \in W_0^{1,p(x)}(\Omega)$ with $|u|_{L^{p^-}(\Omega)} < r$ for some $r \in \mathbb{R}^+$.

Proof From (3.7) it follows that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, s)| \leq C_\varepsilon + \varepsilon |s|^{p(x)-1}. \tag{3.8}$$

Then, according to Theorem 1.16 of [7], f induces a Nemytsky operator on $X = L^{p^-}(\Omega)$, still denoted by f . Setting $T(u) = Kf(u)$, according to Proposition 2.3, $T \in C(X, X)$ is compact and (1.1) can be written in the form $u = Tu$, $u \in X$. Let us show that there exists $r > 0$ such that the homotopy $h(t, u) = u - tT(u)$ is admissible in $B_r = \{u \in X : |u|_{L^{p^-}(\Omega)} < r\}$. Otherwise, there exist $u_n \in X$, with $|u_n|_{L^{p^-}(\Omega)} \rightarrow +\infty$, and $t_n \in [0, 1]$ such that $u_n = t_n T(u_n)$. This is equivalent to $-\operatorname{div}(|\nabla u_n|^{p(x)-2} \nabla u_n) = t_n f(x, u_n)$, with $u_n \in W_0^{1,p(x)}(\Omega)$. Taking u_n as a test function, using (3.8) and the fact that $t_n \leq 1$, we get

$$\int_{\Omega} |\nabla u_n|^{p(x)} dx \leq t_n \int_{\Omega} |f(x, u_n) u_n| dx \leq C_\varepsilon \int_{\Omega} |u_n| dx + \varepsilon \int_{\Omega} |u_n|^{p^-} dx.$$

Case 1. $\|u\| \geq 1$.

In this case, using the Hölder and Poincaré inequality, we deduce

$$c_0^{p^-} |u_n|_{L^{p^-}(\Omega)}^{p^-} \leq C_\varepsilon |u_n|_{L^{p^-}(\Omega)} + \varepsilon |u_n|_{L^{p^-}(\Omega)}^{p^-},$$

where c_0 is the embedding constant of $W_0^{1,p(x)} \hookrightarrow L^{p^-}(\Omega)$. If we take ε such that $\varepsilon < c_0^{p^-}$, this equation implies that $\|u_n\|_{L^{p^-}(\Omega)} \leq C_1$, for some $C_1 > 0$, a contradiction.

Case 2. $\|u\| \leq 1$.

In this case, using the Hölder and Poincaré inequalities, we deduce

$$c_0^{p^+} |u_n|_{L^{p^-}(\Omega)}^{p^+} \leq C_\varepsilon |u_n|_{L^{p^-}(\Omega)} + \varepsilon |u_n|_{L^{p^-}(\Omega)}^{p^-} \leq C_\varepsilon |u_n|_{L^{p^-}(\Omega)} + \varepsilon |u_n|_{L^{p^-}(\Omega)}^{p^+}.$$

If we take ε such that $\varepsilon < c_0^{p^+}$, this equation implies that $\|u_n\|_{L^{p^-}(\Omega)} \leq C_2$, for some $C_2 > 0$, a contradiction again.

Thus the homotopy $h(t, u) = u - tT(u)$ is admissible on the ball B_r . Using the homotopy invariance, it follows that $\operatorname{deg}(I - T, B_r, 0) = \operatorname{deg}(I, B_r, 0) = 1$, and hence there exists $u \in B_r$ such that $u = T(u)$, giving rise to a solution of problem (1.1). \square

Remark 3.1 We note that the fact that the homotopy $h(t, u) = u - tT(u)$ is admissible in B_r implies $|u|_{L^{p^-}(\Omega)} \neq r$ for any solution of (3.2).

Using Theorem 3.1, Theorem 3.2 and Remark 3.1, we easily get:

Corollary 3.2 *Assume $p(x) \in C_+(\overline{\Omega})$ and f satisfies (3.7), then the boundary value problem (1.1) has at least one solution $u \in W_0^{1,p(x)}(\Omega)$ with $|u|_{L^{p^-}(\Omega)} \leq r$ for some $r > 0$.*

4 Existence and uniqueness via the Galerkin method

In this section, we shall use the Brouwer fixed point theorem and the Galerkin method to prove the existence of a solution for problem (1.1).

Theorem 4.1 *Assume that there exist constant $a > 0$ and $\alpha(x) \in C_+(\overline{\Omega})$ with $\alpha^+ < p^-$ such that*

$$f(x, u)u \leq a(1 + |u|^{\alpha(x)}), \quad \forall x \in \Omega, u \in \mathbb{R} \tag{4.1}$$

with

$$a < \frac{1}{C^{\alpha^+}}, \tag{4.2}$$

where C is the embedding constant of $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$. Then problem (1.1) has at least one weak solution. Besides, any solution u of (1.1) satisfies the estimate

$$\|u\| \leq R_1 := \max \left\{ \left(\frac{a|\Omega|}{1 - C^{\alpha^+}} \right), 1 \right\}. \tag{4.3}$$

Proof Because $W_0^{1,p(x)}(\Omega)$ is a reflexive and separable Banach space, there exist $\{e_j\} \subset W_0^{1,p(x)}(\Omega)$ and $\{e_j^*\} \subset (W_0^{1,p(x)}(\Omega))^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \dots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, let us put

$$V_n = \text{Span}\{e_1, \dots, e_n\}.$$

Then V_n is isometric to \mathbb{R}^n . In fact, each $v \in V_n$ is uniquely associated to $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ by the relation $v = \sum_{k=1}^n \eta_k e_k$. We search for solutions $u_n \in V_n$ of the approximate problem

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla e_k \, dx = \int_{\Omega} f(x, u_n) e_k \, dx, \quad k = 1, 2, \dots, n. \tag{4.4}$$

To solve this algebraic system, we define the operator $P_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$(P_n u)_k = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla e_k \, dx - \int_{\Omega} f(x, u) e_k \, dx, \quad u \in V_n.$$

We note that P_n is continuous from the continuity of $f(x, u)$ with respect to u . Therefore, we can use the following form of the Brouwer fixed point theorem: if there exists $R > 0$ such that $\langle P_n u, u \rangle \geq 0$ whenever $\|u\| = R$, then P_n has a root u satisfying $\|u\| \leq R$ (see,

e.g., [25]). From (4.1), Proposition 2.1 and the Poincaré inequality, we have for $u \in V_n$ with $\|u\| \geq 1$

$$\begin{aligned} \langle P_n u, u \rangle &\geq \|u\|^{p^-} - \int_{\Omega} f(x, u)u \, dx \\ &\geq \|u\|^{p^-} - aC^{\alpha^+} \|u\|^{\alpha^+} - a|\Omega|. \end{aligned}$$

This shows, from (4.2), the existence of $R > 1$, depending only on C , a and $|\Omega|$, such that $\langle P_n u, u \rangle \geq 0$ if $\|u\| = R$. Then system (4.4) has a solution $u_n \in V_n$ satisfying

$$\|u_n\| \leq R, \quad \forall n \in \mathbb{N}.$$

From this estimate, going to a subsequence if necessary, there is u such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega).$$

Besides, since $W_0^{1,p(x)}(\Omega) \rightarrow L^{\alpha(x)}(\Omega)$ compactly and the Nemytsky map N_f is continuous from $L^{\alpha(x)}(\Omega)$ to $L^{\alpha'(x)}(\Omega)$ (see [7]). Then fixing k in (4.4) and letting $n \rightarrow +\infty$, we conclude that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla e_k \, dx = \int_{\Omega} f(x, u) e_k \, dx, \quad k = 1, 2, 3, \dots \tag{4.5}$$

From the completeness of $\{e_k\}$, identity (4.5) holds with e_k replaced by any $\varphi \in W_0^{1,p(x)}(\Omega)$, we get

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} f(x, u) \varphi \, dx,$$

which shows that u is in fact a solution of problem (1.1). Finally, if u is any solution of problem (1.1), then $\int_{\Omega} |\nabla u|^{p(x)} \, dx = \int_{\Omega} f(x, u)u \, dx$. Therefore, either $\|u\| \leq 1$ or

$$\|u\|^{p^-} \leq aC^{\alpha^+} \|u\|^{\alpha^+} + a|\Omega| \leq aC^{p^-} \|u\|^{\alpha^+} + a|\Omega| \tag{4.6}$$

and (4.3) follows. □

Theorem 4.2 *Let the assumptions of Theorem 4.1 hold, with (4.1) replaced by*

$$(f(x, u) - f(x, v))(u - v) \leq 0, \quad \forall x \in \Omega, \forall u, v \in \mathbb{R}. \tag{4.7}$$

Then problem (1.1) has exactly one solution.

Proof Taking $v = 0$ in (4.7), we get

$$f(x, u)u \leq f(x, 0)u \leq \max_{x \in \overline{\Omega}} |f(x, 0)| + \max_{x \in \overline{\Omega}} |f(x, 0)| |u|^{\alpha(x)}. \tag{4.8}$$

Setting $a := \max_{x \in \overline{\Omega}} |f(x, 0)|$ in (4.8), we get (4.1). Hence, the existence part follows from Theorem 4.1. Now let u and v be two solutions of problem (1.1). Putting $w = u - v$, by (4.7)

and Theorem 3.1 of [19], we have

$$0 \leq \int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \nabla w \, dx = \int_{\Omega} (f(x, u) - f(x, v)) w \, dx \leq 0.$$

We conclude that $w \equiv 0$, and hence $u = v$. □

Competing interests

We declare that we have no competing interests.

Authors' contributions

GD conceived of the study, participated in its design and coordination, and helped to draft the manuscript. SY participated in the design of the study. All authors read and approved the final manuscript.

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable suggestions. Research supported by the NSFC (No. 11261052, No. 11126296).

Received: 8 June 2012 Accepted: 22 August 2012 Published: 7 September 2012

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doi:10.1186/1687-2770-2012-99

Cite this article as: Yang and Dai: Existence results for a variable exponent elliptic problem via topological method. *Boundary Value Problems* 2012 **2012**:99.