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The existence of global weak solutions for a weakly dissipative Camassa-Holm equation in $H^1(R)$

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Abstract

The existence of global weak solutions to the Cauchy problem for a weakly dissipative Camassa-Holm equation is established in the space $C([0, \infty) \times R) \cap L^\infty([0, \infty); H^1(R))$ under the assumption that the initial value $u_0(x)$ only belongs to the space $H^1(R)$. The limit of viscous approximations, a one-sided super bound estimate and a space-time higher-norm estimate for the equation are established to prove the existence of the global weak solution.
MSC: 35G25; 35L05

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1 Introduction

In this work, we investigate the Cauchy problem for the nonlinear model

$$u_t - u_{txx} + \partial_x f(u) = 2u_x u_{xx} + u u_{xxx} - \lambda u^{2N+1} + \beta u^{2m} u_{xx}, \qquad (1)$$

where $\lambda \geq 0$, $\beta \geq 0$, $f(u)$ is a polynomial with order n, N and m are nonnegative integers. When $f(u) = 2ku + \frac{3}{2}u^2$, $\lambda = 0$, $\beta = 0$, Eq. (1) is the standard Camassa-Holm equation [1–3]. In fact, the nonlinear term $-\lambda u^{2N+1} + \beta u^{2m} u_{xx}$ can be regarded as a weakly dissipative term for the Camassa-Holm model (see [4, 5]). Here we coin (1) a weakly dissipative Camassa-Holm equation.

To link with previous works, we review several works on global weak solutions for the Camassa-Holm and Degasperis-Procesi equations. The existence and uniqueness results for global weak solutions of the standard Camassa-Holm equation have been proved by Constantin and Escher [6], Constantin and Molinet [7] and Danchin [8, 9] under the sign condition imposing on the initial value. Xin and Zhang [10] established the global existence of a weak solution for the Camassa-Holm equation in the energy space $H^1(R)$ without imposing the sign conditions on the initial value, and the uniqueness of the weak solution was obtained under certain conditions on the solution [11]. Under the sign condition for the initial value, Yin and Lai [12] proved the existence and uniqueness results of a global weak solution for a nonlinear shallow water equation, which includes the famous Camassa-Holm and Degasperis-Procesi equations as special cases. Lai and Wu [13] obtained the existence of a local weak solution for Eq. (1) in the lower-order Sobolev space $H^s(R)$ with $1 \leq s \leq \frac{3}{2}$. For other meaningful methods to handle the problems relating to dy-

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namic properties of the Camassa-Holm equation and other partial differential equations, the reader is referred to [14–19]. Coclite *et al.* [20] used the analysis presented in [10, 11] and investigated global weak solutions for a generalized hyperelastic-rod wave equation (or a generalized Camassa-Holm equation), namely, $\lambda = 0$, $\beta = 0$ in Eq. (1). The existence of a strongly continuous semigroup of global weak solutions for the generalized hyperelastic-rod equation with any initial value in the space $H^1(R)$ was established in [20]. Up to now, the existence result of the global weak solution for the weakly dissipative Camassa-Holm equation (1) has not been found in the literature. This constitutes the motivation of this work.

The objective of this work is to study the existence of global weak solutions for the Eq. (1) in the space $C([0, \infty) \times R) \cap L^\infty([0, \infty); H^1(R))$ under the assumption $u_0(x) \in H^1(R)$. The key elements in our analysis include some new *a priori* one-sided upper bound and space-time higher-norm estimates on the first-order derivatives of the solution. Also, the limit of viscous approximations for the equation is used to establish the existence of the global weak solution. Here we should mention that the approaches used in this work come from Xin and Zhang [10] and Coclite *et al.* [20].

The rest of this paper is as follows. The main result is given in Section 2. In Section 3, we present a viscous problem of Eq. (1) and give a corresponding well-posedness result. An upper bound, a higher integrability estimate and some basic compactness properties for the viscous approximations are also established in Section 3. Strong compactness of the derivative of the viscous approximations is obtained in Section 4, where the main result for the existence of Eq. (1) is proved.

2 Main result

Consider the Cauchy problem for Eq. (1)

$$
\begin{cases}
u_t - u_{txx} + \partial_x f(u) = 2u_x u_{xx} + u u_{xxx} - \lambda u^{2N+1} + \beta u^{2m} u_{xx}, \\
u(0, x) = u_0(x),
\end{cases}
\tag{2}
$$

which is equivalent to

$$
\begin{cases}
u_t + u u_x + \frac{\partial P}{\partial x} = 0, \\
\frac{\partial P}{\partial x} = \Lambda^{-2} \partial_x \left[f(u) + \frac{1}{2}(u_x^2 - u^2) - \beta u^{2m} u_x \right] + \Lambda^{-2} \left[\lambda u^{2N+1} + 2m\beta u^{2m-1} u_x^2 \right], \\
u(0, x) = u_0(x),
\end{cases}
\tag{3}
$$

where the operator $\Lambda^2 = 1 - \frac{\partial^2}{\partial x^2}$. For a fixed $1 \leq p_0 < \infty$, one has

$$
\Lambda^{-2} g(x) = \frac{1}{2} \int_R e^{-|x-y|} g(y) \, dy \quad \text{for } g(x) \in L^{p_0}(R), 1 < p_0 < \infty.
$$

In fact, as proved in [13], problem (2) satisfies the following conservation law:

$$
\int_R \left(u^2 + u_x^2 \right) dx + 2\lambda \int_0^t \int_R u^{2N+2} \, dx \, dt + 2\beta(2m+1) \int_0^t \int_R u^{2m} u_x^2 \, dx \, dt
$$
$$
= \int_R \left(u_0^2 + u_{0x}^2 \right) dx.
\tag{4}
$$

Now we introduce the definition of a weak solution to Cauchy problem (2) or (3).

Definition 1 A continuous function $u : [0, \infty) \times R \to R$ is said to be a global weak solution to Cauchy problem (3) if

(i) $u \in C([0, \infty) \times R) \cap L^\infty([0, \infty); H^1(R))$;

(ii) $\|u(t, \cdot)\|_{H^1(R)} \leq \|u_0\|_{H^1(R)}$;

(iii) $u = u(t, x)$ satisfies (3) in the sense of distributions and takes on the initial value pointwise.

The main result of this paper is stated as follows.

Theorem 1 Assume $u_0(x) \in H^1(R)$. Then Cauchy problem (2) or (3) has a global weak solution $u(t, x)$ in the sense of Definition 1. Furthermore, the weak solution satisfies the following properties.

(a) There exists a positive constant c_0 depending on $\|u_0\|_{H^1(R)}$ and the coefficients of Eq. (1) such that the following one-sided L^∞ norm estimate on the first-order spatial derivative holds:

$$\frac{\partial u(t, x)}{\partial x} \leq \frac{4}{t} + c_0, \quad for\ (t, x) \in [0, \infty) \times R. \tag{5}$$

(b) Let $0 < \gamma < 1$, $T > 0$ and $a, b \in R$, $a < b$. Then there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b and the coefficients of Eq. (1) such that the following space higher integrability estimate holds:

$$\int_0^t \int_a^b \left| \frac{\partial u(t, x)}{\partial x} \right|^{2+\gamma} dx \leq c_1. \tag{6}$$

3 Viscous approximations

Defining

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{7}$$

and setting the mollifier $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}} \phi(\varepsilon^{-\frac{1}{4}} x)$ with $0 < \varepsilon < \frac{1}{4}$ and $u_{\varepsilon,0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon,0} \in C^\infty$ for any $u_0 \in H^s$, $s > 0$ (see Lai and Wu [13]). In fact, choosing the mollifier properly, we have

$$\|u_{\varepsilon,0}\|_{H^1(R)} \leq \|u_0\|_{H^1(R)} \quad \text{and} \quad u_{\varepsilon,0} \to u_0 \quad \text{in } H^1(R). \tag{8}$$

The existence of a weak solution to Cauchy problem (3) will be established by proving the compactness of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon > 0}$ solving the following viscous problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial P_\varepsilon}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}, \\ \frac{\partial P_\varepsilon}{\partial x} = \Lambda^{-2} \partial_x [f(u_\varepsilon) - \frac{1}{2} u_\varepsilon^2 + \frac{1}{2}(\frac{\partial u_\varepsilon}{\partial x})^2 - \beta u_\varepsilon^{2m} \frac{\partial u_\varepsilon}{\partial x}] \\ \qquad + \lambda \Lambda^{-2}(u_\varepsilon)^{2N+1} + 2m\beta \Lambda^{-2}[u_\varepsilon^{2m-1}(\frac{\partial u_\varepsilon}{\partial x})^2], \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x). \end{cases} \tag{9}$$

Now we start our analysis by establishing the following well-posedness result for problem (9).

Lemma 3.1 *Provided that $u_0 \in H^1(R)$, for any $\sigma \geq 3$, there exists a unique solution $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ to Cauchy problem (9). Moreover, for any $t > 0$, it holds that*

$$\int_R \left(u_\varepsilon^2 + \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 \right) dx + \int_0^t \int_R \left(2\lambda u_\varepsilon^{2N+2} + 2\beta(2m+1) u_\varepsilon^{2m} \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 \right) dx\,dt$$
$$+ 2\varepsilon \int_0^t \int_R \left(\left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2} \right)^2 \right)(s,x)\,dx\,ds = \|u_{\varepsilon,0}\|_{H^1(R)}^2, \tag{10}$$

or

$$\|u_\varepsilon(t,\cdot)\|_{H^1(R)}^2 + 2\varepsilon \int_0^t \left\| \frac{\partial u_\varepsilon}{\partial x}(s,\cdot) \right\|_{H^1(R)}^2 ds$$
$$+ \int_0^t \int_R \left(2\lambda u^{2N+2} + 2\beta(2m+1) u_\varepsilon^{2m} \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 \right) dx\,dt = \|u_{\varepsilon,0}\|_{H^1(R)}^2. \tag{11}$$

Proof For any $\sigma \geq 3$ and $u_0 \in H^1(R)$, we have $u_{\varepsilon,0} \in C([0,\infty); H^\sigma(R))$. From Theorem 2.3 in [21], we conclude that problem (9) has a unique solution $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ for an arbitrary $\sigma > 3$.

We know that the first equation in system (9) is equivalent to the form

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\partial^3 u_\varepsilon}{\partial t x^2} + \frac{\partial f(u_\varepsilon)}{\partial x} = 2 \frac{\partial u_\varepsilon}{\partial x} \frac{\partial^2 u_\varepsilon}{\partial x^2} + u_\varepsilon \frac{\partial^3 u_\varepsilon}{\partial x^3} - \lambda u_\varepsilon^{2N+1} + \beta u_\varepsilon^{2m} \frac{\partial^2 u_\varepsilon}{\partial x^2}$$
$$+ \varepsilon \left(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^4 u_\varepsilon}{\partial x^4} \right), \tag{12}$$

from which we derive that

$$\frac{1}{2} \frac{d}{dt} \int_R \left(u_\varepsilon^2 + \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 \right) dx + \lambda \int_R u_\varepsilon^{2N+1}\,dx + \beta(2m+1) \int_R u_\varepsilon^{2m} \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 dx$$
$$+ \varepsilon \int_R \left(\left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2} \right)^2 \right) dx = 0, \tag{13}$$

which completes the proof. □

From Lemma 3.1 and (8), we have

$$\|u_\varepsilon\|_{L^\infty(R)} \leq \|u_\varepsilon\|_{H^1(R)} \leq \|u_{\varepsilon,0}\|_{H^1(R)} \leq \|u_0\|_{H^1(R)}. \tag{14}$$

Differentiating the first equation of problem (9) with respect to x and writing $\frac{\partial u_\varepsilon}{\partial x} = q_\varepsilon$, we obtain

$$\frac{\partial q_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial q_\varepsilon}{\partial x} - \varepsilon \frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{1}{2} q_\varepsilon^2 + \beta(u_\varepsilon)^{2m} q_\varepsilon$$
$$= f(u_\varepsilon) - \frac{1}{2} u_\varepsilon^2 - \Lambda^{-2} \left[f(u_\varepsilon) - \frac{1}{2}(u_\varepsilon^2 - q^2) - \beta(u_\varepsilon)^{2m} q_\varepsilon \right]$$

$$+ \lambda \left(u_\varepsilon^{2N+1} \right)_x + 2m\beta \partial_x \left[(u_\varepsilon)^{2m-1} q_\varepsilon^2 \right] \Bigg]$$
$$= Q_\varepsilon(t,x). \tag{15}$$

Lemma 3.2 *Let $0 < \gamma < 1$, $T > 0$ and $a, b \in R$, $a < b$. Then there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, γ, T, a, b and the coefficients of Eq.* (1), *but independent of ε, such that the space higher integrability estimate holds*

$$\int_0^T \int_a^b \left| \frac{\partial u_\varepsilon(t,x)}{\partial x} \right|^{2+\gamma} dx \le c_1, \tag{16}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of problem (9).

The proof is similar to that of Proposition 3.2 presented in Xin and Zhang [10] (also see Coclite *et al.* [20]). Here we omit it.

Lemma 3.3 *There exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of Eq.* (1) *such that*

$$\left\| Q_\varepsilon(t, \cdot) \right\|_{L^\infty(R)} \le C, \qquad \left\| Q_\varepsilon(t, \cdot) \right\|_{L^1(R)} \le C, \qquad \left\| Q_\varepsilon(t, \cdot) \right\|_{L^2(R)} \le C, \tag{17}$$

$$\left\| \frac{\partial P_\varepsilon(t, \cdot)}{\partial x} \right\|_{L^\infty(R)} \le C, \qquad \left\| \frac{\partial P_\varepsilon(t, \cdot)}{\partial x} \right\|_{L^1(R)} \le C, \qquad \left\| \frac{\partial P_\varepsilon(t, \cdot)}{\partial x} \right\|_{L^2(R)} \le C, \tag{18}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of system (9).

Due to strong similarities with the proof of Lemma 5.1 presented in Coclite *et al.* [20], we do not prove Lemma 3.3 here.

Lemma 3.4 *Assume that $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of* (9). *For an arbitrary $T > 0$, there exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of Eq.* (1) *such that the following one-sided L^∞ norm estimate on the first-order spatial derivative holds*:

$$\frac{\partial u_\varepsilon(t,x)}{\partial x} \le \frac{4}{t} + C \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{19}$$

Proof From (15) and Lemma 3.3, we know that there exists a positive constant C depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of Eq. (1) such that $\|Q_\varepsilon(t,x)\|_{L^\infty(R)} \le C$. Therefore,

$$\frac{\partial q_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial q_\varepsilon}{\partial x} + \frac{1}{2} q_\varepsilon^2 + \beta (u_\varepsilon)^{2m} q_\varepsilon = Q_\varepsilon(t,x) \le C. \tag{20}$$

Let $f = f(t)$ be the solution of

$$\frac{df}{dt} + \frac{1}{2} f^2 + \beta \left(u^* \right)^{2m} f = C, \quad t > 0, \qquad f(0) = \left\| \frac{\partial u_{\varepsilon,0}}{\partial x} \right\|_{L^\infty(R)}, \tag{21}$$

where u_ε^* is the value of $u_\varepsilon(t,x)$ when $\sup_{x \in R} q_\varepsilon(t,x) = f(t)$. From the comparison principle for parabolic equations, we get

$$q_\varepsilon(t,x) \le f(t). \tag{22}$$

Using (14) and $-\beta(u^*)^{2m}f \le \rho^2 f^2 + \frac{1}{4\rho^2}\beta^2(u^*)^{4m}$, we derive that

$$\frac{df}{dt} = C - \frac{1}{2}f^2 - \beta(u^*)^{2m}f \le C - \frac{1}{2}f^2 + \rho^2 f^2 + \frac{1}{4\rho^2}\beta^2(u^*)^{4m}$$

$$\le C - \frac{1}{4}f^2 + C_1, \tag{23}$$

where $\|\frac{1}{4\rho^2}\beta^2(u^*)^{4m}\| \le C_1$ and $\rho = \frac{1}{2}$. Setting $M_0 = C + C_1$, we obtain

$$\frac{df}{dt} + \frac{1}{4}f^2 \le M_0. \tag{24}$$

Letting $F(t) = \frac{4}{t} + 2\sqrt{M_0}$, we have $\frac{dF(t)}{dt} + \frac{1}{4}F^2(t) - M_0 = \frac{4\sqrt{M_0}}{t} > 0$. From the comparison principle for ordinary differential equations, we get $f(t) \le F(t)$ for all $t > 0$. Therefore, by this and (22), the estimate (19) is proved. □

Lemma 3.5 *For $u_0 \in H^1(R)$, there exists a sequence $\{\varepsilon_j\}_{j \in N}$ tending to zero and a function $u \in L^\infty([0,\infty); H^1(R)) \cap H^1([0,T] \times R)$ such that, for each $T \ge 0$, it holds that*

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in } H^1([0,T] \times R), \text{ for each } T \ge 0, \tag{25}$$

$$u_{\varepsilon_j} \to u \quad \text{in } L^\infty_{\text{loc}}([0,\infty) \times R), \tag{26}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (9).

Lemma 3.6 *There exists a sequence $\{\varepsilon_j\}_{j \in N}$ tending to zero and a function $Q \in L^\infty([0,\infty) \times R)$ such that for each $1 < p < \infty$,*

$$Q_{\varepsilon_j} \to Q \quad \text{strongly in } L^p_{\text{loc}}([0,\infty) \times R). \tag{27}$$

The proofs of Lemmas 3.5 and 3.6 are similar to those of Lemmas 5.2 and 5.3 in [20]. Here we omit their proofs.

Throughout this paper, we use overbars to denote weak limits (the space in which these weak limits are taken is $L^r_{\text{loc}}([0,\infty) \times R)$ with $1 < r < \frac{3}{2}$).

Lemma 3.7 *There exists a sequence $\{\varepsilon_j\}_{j \in N}$ tending to zero and two functions $q \in L^p_{\text{loc}}([0,\infty) \times R), \overline{q^2} \in L^r_{\text{loc}}([0,\infty) \times R)$ such that*

$$q_{\varepsilon_j} \rightharpoonup q \quad \text{in } L^p_{\text{loc}}([0,\infty) \times R), \qquad q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \text{in } L^\infty_{\text{loc}}([0,\infty); L^2(R)), \tag{28}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in } L^r_{\text{loc}}([0,\infty) \times R), \tag{29}$$

for each $1 < p < 3$ and $1 < r < \frac{3}{2}$. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x) \quad \text{for almost every } (t,x) \in [0,\infty) \times R \tag{30}$$

and

$$\frac{\partial u}{\partial x} = q \quad \text{in the sense of distributions on } [0,\infty) \times R. \tag{31}$$

Proof (28) and (29) are a direct consequence of Lemmas 3.1 and 3.2. Inequality (30) is valid because of the weak convergence in (29). Finally, (31) is a consequence of the definition of q_ε, Lemma 3.5 and (28). $\qquad\square$

In the following, for notational convenience, we replace the sequence $\{u_{\varepsilon_j}\}_{j\in N}$, $\{q_{\varepsilon_j}\}_{j\in N}$ and $\{Q_{\varepsilon_j}\}_{j\in N}$ by $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$ and $\{Q_\varepsilon\}_{\varepsilon>0}$, respectively.

Using (28), we conclude that for any convex function $\eta \in C^1(R)$ with η' being bounded and Lipschitz continuous on R and for any $1 < p < 3$, we get

$$\eta(q_\varepsilon) \rightharpoonup \overline{\eta(q)} \quad in\ L^p_{\mathrm{loc}}\big([0,\infty)\times R\big), \tag{32}$$

$$\eta(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\eta(q)} \quad in\ L^\infty_{\mathrm{loc}}\big([0,\infty);L^2(R)\big). \tag{33}$$

Multiplying Eq. (15) by $\eta'(q_\varepsilon)$ yields

$$\frac{\partial}{\partial t}\eta(q_\varepsilon) + \frac{\partial}{\partial x}\big(u_\varepsilon\eta(q_\varepsilon)\big) - \varepsilon\frac{\partial^2}{\partial x^2}\eta(q_\varepsilon) + \varepsilon\eta''(q_\varepsilon)\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2$$

$$= q_\varepsilon\eta(q_\varepsilon) - \frac{1}{2}\eta'(q_\varepsilon)q_\varepsilon^2 - \beta(u_\varepsilon)^{2m}q_\varepsilon\eta'(q_\varepsilon) + Q_\varepsilon(t,x)\eta'(q_\varepsilon). \tag{34}$$

Lemma 3.8 *For any convex $\eta \in C^1(R)$ with η' being bounded and Lipschitz continuous on R, it holds that*

$$\overline{\frac{\partial\eta(q)}{\partial t}} + \frac{\partial}{\partial x}\big(u\overline{\eta(q)}\big) \leq \overline{q\eta(q)} - \frac{1}{2}\overline{\eta'(q)q^2} - \beta u^{2m}\overline{q\eta'(q)} + Q(t,x)\overline{\eta'(q)} \tag{35}$$

in the sense of distributions on $[0,\infty)\times R$. Here $\overline{q\eta(q)}$ and $\overline{\eta'(q)q^2}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$ and $q_\varepsilon^2\eta'(q_\varepsilon)$ in $L^r_{\mathrm{loc}}([0,\infty)\times R)$, $1 < r < \frac{3}{2}$, respectively.

Proof In (34), by the convexity of η, (14), Lemmas 3.5, 3.6 and 3.7, taking limit for $\varepsilon \to 0$ gives rise to the desired result. $\qquad\square$

Remark 3.9 From (28) and (29), we know that

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \qquad q^2 = (q_+)^2 + (q_-)^2, \qquad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2} \tag{36}$$

almost everywhere in $[0,\infty)\times R$, where $\xi_+ := \xi_{\chi[0,+\infty)}(\xi)$, $\xi_- := \xi_{\chi(-\infty,0]}(\xi)$ for $\xi \in R$. From Lemma 3.4 and (28), we have

$$q_\varepsilon(t,x), \qquad q(t,x) \leq \frac{4}{t} + C \quad for\ t > 0, x \in R, \tag{37}$$

where C is a constant depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of Eq. (1).

Lemma 3.10 *In the sense of distributions on $[0,\infty)\times R$, it holds that*

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(uq) = \frac{1}{2}\overline{q^2} - \beta u^{2m}q + Q(t,x). \tag{38}$$

Proof Using (15), Lemmas 3.5 and 3.6, (28), (29) and (31), the conclusion (38) holds by taking limit for $\varepsilon \to 0$ in (15). □

The next lemma contains a generalized formulation of (38).

Lemma 3.11 *For any $\eta \in C^1(R)$ with $\eta' \in L^\infty(R)$, it holds that*

$$\frac{\partial \eta(q)}{\partial t} + \frac{\partial}{\partial x}\big(u\eta(q)\big) = q\eta(q) + \left(\frac{1}{2}\overline{q^2} - q^2\right)\eta'(q) - \beta u^{2m}q\eta'(q) + Q(t,x)\eta'(q) \tag{39}$$

in the sense of distributions on $[0,\infty) \times R$.

Proof Let $\{\omega_\delta\}_\delta$ be a family of mollifiers defined on R. Denote $q_\delta(t,x) := (q(t,\cdot) \star \omega_\delta)(x)$, where the $\star$ is the convolution with respect to x variable. Multiplying (38) by $\eta'(q_\delta)$ yields

$$\begin{aligned}
\frac{\partial \eta(q_\delta)}{\partial t} &= \eta'(q_\delta)\frac{\partial q_\delta}{\partial t} \\
&= \eta'(q_\delta)\left[\frac{1}{2}\overline{q^2} \star \omega_\delta - \beta u^{2m}q_\delta + Q(t,x) \star \omega_\delta - q^2 \star \omega_\delta - u\frac{\partial q}{\partial x} \star \omega_\delta\right]
\end{aligned} \tag{40}$$

and

$$\frac{\partial}{\partial x}\big(u\eta(q_\delta)\big) = q\eta(q_\delta) + u\eta'(q_\delta)\left(\frac{\partial q_\delta}{\partial x}\right). \tag{41}$$

Using the boundedness of η, η' and letting $\delta \to 0$ in the above two equations, we obtain (39). □

4 Strong convergence of q_ε

Now, we will prove the strong convergence result, *i.e.*,

$$\partial_x u_\varepsilon \to \partial_x u \quad \text{as } \varepsilon \to 0 \text{ in } L^2_{\text{loc}}\big([0,\infty) \times R\big), \tag{42}$$

which is one of key statements to derive that $u(t,x)$ is a global weak solution required in Theorem 1.

Lemma 4.1 *Assume $u_0 \in H^1(R)$. It holds that*

$$\lim_{t \to 0}\int_R q^2(t,x)\,dx = \lim_{t \to 0}\int_R \overline{q^2}(t,x)\,dx = \int_R \left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{43}$$

Lemma 4.2 *If $u_0 \in H^1(R)$, for each $M > 0$, it holds that*

$$\lim_{t \to 0}\int_R \big(\overline{\eta_M^\pm(q)}(t,x) - \eta_M^\pm\big(q(t,x)\big)\big)\,dx = 0, \tag{44}$$

where

$$\eta_M(\xi) := \begin{cases} \frac{1}{2}\xi^2 & \text{if } |\xi| \leq M, \\ M|\xi| - \frac{1}{2}M^2 & \text{if } |\xi| > M, \end{cases} \tag{45}$$

and $\eta_M^+(\xi) := \eta_M(\xi)\chi_{[0,+\infty)}(\xi)$, $\eta_M^-(\xi) := \eta_M(\xi)\chi_{(-\infty,0]}(\xi)$, $\xi \in R$.

Lemma 4.3 *Let $M > 0$. Then for each $\xi \in R$,*

$$
\begin{cases}
\eta_M(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}(M - |\xi|)^2 \chi_{(-\infty,-M)\cap(M,\infty)}(\xi), \\
\eta_M'(\xi)\xi = \xi + (M - |\xi|)\operatorname{sign}(\xi)\chi_{(-\infty,-M)\cap(M,\infty)}(\xi), \\
\eta_M^+(\xi) = \frac{1}{2}(\xi_+)^2 - \frac{1}{2}(M - \xi)^2 \chi_{(M,\infty)}(\xi), \\
(\eta_M^+)'(\xi) = \xi_+ + (M - \xi)\chi_{(M,\infty)}(\xi), \\
\eta_M^-(\xi) = \frac{1}{2}(\xi_-)^2 - \frac{1}{2}(M + \xi)^2 \chi_{(-\infty,-M)}(\xi), \\
(\eta_M^-)'(\xi) = \xi_- - (M + \xi)\chi_{(-\infty,-M)}(\xi).
\end{cases}
\tag{46}
$$

The proofs of Lemmas 4.1, 4.2 and 4.3 can be found in [10] or [20].

Lemma 4.4 *Assume $u_0 \in H^1(R)$. Then for almost all $t > 0$,*

$$
\frac{1}{2}\int_R \big(\overline{(q_+)^2} - q_+^2\big)(t,x)\,dx \leq \int_0^t \int_R Q(s,x)\big[\overline{q_+}(s,x) - q_+(s,x)\big]\,ds\,dx.
\tag{47}
$$

Lemma 4.5 *For any $t > 0$, $M > 0$ and $u_0 \in H^1(R)$, it holds that*

$$
\int_R \big(\overline{\eta_M^-(q)} - \eta_M^-(q)\big)(t,x)\,dx
$$
$$
\leq \frac{M^2}{2}\int_0^t \int_R u\overline{(M + q)\chi_{(-\infty,-M)}(q)}\,ds\,dx
$$
$$
- \frac{M^2}{2}\int_0^t \int_R u(M + q)\chi_{(-\infty,-M)}(q)\,ds\,dx + M\int_0^t \int_R u\big[\overline{\eta_M^-(q)} - \eta_M^-(q)\big]\,ds\,dx
$$
$$
+ \frac{M}{2}\int_0^t \int_R u\big(\overline{q_+^2} - q_+^2\big)\,ds\,dx + \int_0^t \int_R Q(t,x)\big(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big)\,ds\,dx.
\tag{48}
$$

We do not provide the proofs of Lemmas 4.4 and 4.5 since they are similar to those of Lemmas 6.4 and 6.5 in Coclite *et al.* [20].

Lemma 4.6 *Assume $u_0 \in H^1(R)$. Then it has*

$$
\overline{q^2} = q^2 \quad \text{almost everywhere in } [0,\infty) \times (-\infty,\infty).
\tag{49}
$$

Proof Applying Lemmas 4.4 and 4.5 gives rise to

$$
\int_R \left(\frac{1}{2}\big[\overline{(q_+)^2} - (q_+)^2\big] + \big[\overline{\eta_M^-} - \eta_M^-\big]\right)(t,x)\,dx
$$
$$
\leq \frac{M^2}{2}\left(\int_0^t \int_R \overline{(M + q)\chi_{(-\infty,-M)}(q)}\,ds\,dx\right.
$$
$$
\left. - \frac{M^2}{2}\int_0^t \int_R (M + q)\chi_{(-\infty,-M)(q)}\,ds\,dx\right)
$$
$$
+ M\int_0^t \int_R \big[\overline{\eta_M^-} - \eta_M^-\big]\,ds\,dx + \frac{M}{2}\int_0^t \int_R \big[\overline{(q_+)^2} - (q_+)^2\big]\,ds\,dx
$$
$$
+ \int_0^t \int_R Q(s,x)\big([\overline{q_+} - q_+] + \big[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big]\big)\,ds\,dx.
\tag{50}
$$

From Lemma 3.6, we know that there exists a constant $L > 0$, depending only on $\|u_0\|_{H^1(R)}$, such that

$$\left\| Q(t,x) \right\|_{L^\infty([0,\infty)\times R)} \le L. \tag{51}$$

By Remark 3.9 and Lemma 4.3, one has

$$q_+ + \left(\eta_M^-\right)'(q) = q - (M+q)\chi_{(-\infty,-M)},$$
$$\overline{q_+} + \overline{\left(\eta_M^-\right)'(q)} = q - \overline{(M+q)\chi_{(-\infty,-M)}}(q). \tag{52}$$

Thus, by the convexity of the map $\xi \to \xi_+ + (\eta_M^-)'(\xi)$, we get

$$0 \le \left[\overline{q_+} - q_+\right] + \left[\overline{\left(\eta_M^-\right)'(q)} - \left(\eta_M^-\right)'(q)\right]$$
$$= (M+q)\chi_{(-\infty,-M)} - \overline{(M+q)\chi_{(-\infty,-M)}}(q). \tag{53}$$

Using (51) derives

$$Q(s,x)\left(\left[\overline{q_+} - q_+\right] + \left[\overline{\left(\eta_M^-\right)'(q)} - \left(\eta_M^-\right)'(q)\right]\right)$$
$$\le -L\left(\overline{(M+q)\chi_{(-\infty,-M)}}(q) - (M+q)\chi_{(-\infty,-M)}\right). \tag{54}$$

Since $\xi \to (M+\xi)\chi_{(-\infty,-M)}$ is concave, choosing M large enough, we have

$$\frac{M^2}{2}\left(\overline{(M+q)\chi_{(-\infty,-M)}}(q) - (M+q)\chi_{(-\infty,-M)}\right)$$
$$\quad + Q(s,x)\left(\left[\overline{q_+} - q_+\right] + \left[\overline{\left(\eta_M^-\right)'(q)} - \left(\eta_M^-\right)'(q)\right]\right)$$
$$\le \left(\frac{M^2}{2} - L\right)\left(\overline{(M+q)\chi_{(-\infty,-M)}}(q) - (M+q)\chi_{(-\infty,-M)}\right) \le 0. \tag{55}$$

Then, from (50) and (55), we have

$$0 \le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,dx$$
$$\le cM \int_0^t \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,ds\,dx. \tag{56}$$

By using the Gronwall inequality, for each $t > 0$, we have

$$0 \le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,dx = 0.$$

By the Fatou lemma, Remark 3.9 and (30), letting $M \to \infty$, we obtain

$$0 \le \int_R \left(\overline{q^2} - q^2\right)(t,x)\,dx = 0 \quad \text{for } t > 0, \tag{57}$$

which completes the proof. □

Proof of the main result Using (8), (10) and Lemma 3.5, we know that the conditions (i) and (ii) in Definition 1 are satisfied. We have to verify (iii). Due to Lemma 4.2 and Lemma 4.6, we have

$$q_\varepsilon \to q \quad \text{in } L^2_{\text{loc}}\big([0,\infty) \times R\big). \tag{58}$$

From Lemma 3.5, (27) and (58), we know that u is a distributional solution to problem (3). In addition, inequalities (5) and (6) are deduced from Lemmas 3.2 and 3.4. The proof of Theorem 1 is completed. □

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